Computer Simulation of Repairable Processes

Dave Trindade, Ph.D.

Senior AMD Fellow
Applied Statistics
Introduction

Computer simulation

- Powerful tool to recreate possible outcomes of random processes
- Allows checking models and analysis procedures without extensive experimentation
- Where theory is limited, may be the only avenue for solving complex problems
- Source of data for training
Outline of Topics

- Discuss a general procedure for simulating data for any distribution
- Apply to example for estimating parameters of Weibull distribution
- Review concepts of repairable systems
  - Renewal processes
  - Non-renewal processes
- Describe simulation procedure for non-renewal model
- Illustrate using power relation model for NHPP
Overview

- Illustrate the value of computer simulation for solving many different types of simple and complex problems in design, manufacturing, analysis, etc.
- Show the power of EXCEL for simulating, analyzing, and modeling random processes
Simulating a Distribution

Generating random events for a specified situation.

Want to produce failure times from a specific distribution.

To simulate any distribution, we will use a special property of the cumulative distribution function (CDF).
Simulating
Random Variables

Let $U$ be a uniform random variable in the interval $(0,1)$.

If $F(t)$ is any CDF with an inverse $t = F^{-1}$, then substituting $U$ for $F$ in the inverse expression generates a random variable $T$ distributed according to $F$. 
Weibull Distribution

Example

The CDF for a Weibull distribution with characteristic life $c$ and shape parameter $m$ is

$$F(t) = 1 - e^{-(t/c)^m}$$

$F(t)$ is the fraction failing in the population by time $t$. $F(t)$ is also the probability of failure by time $t$.

The inverse expression is:

$$t = F^{-1} = c[-\ln(1 - F)]^{1/m}$$
Simulating Weibull Times to Failure

Substitute the unit uniform random variable $U$ in the inverse expression to get:

$$T = F^{-1}(U) = c \left[- \ln (1 - U)\right]^{1/m}$$

Since both $1-U$ and $U$ are a uniformly distributed variate in the interval $(0,1)$, simplify the inverse expression as:

$$T = c \left[- \ln U \right]^{1/m}$$

Each $U$ value produces a time to failure $T$ from a Weibull distribution with parameters $c$ and $m$. 
Weibull Simulation
Example in EXCEL

Estimating the parameters $c$ and $m$ of a Weibull distribution is not simple, even for complete data. For MLE’s, iterative numerical method are required.

Let’s investigate the suitability of the following simple procedure for estimating the parameters.

Write the 25th and 75th percentiles for a Weibull distribution as:

$$t_{25} = c\left[-\ln(1-0.25)\right]^{1/m} = c\left[\ln(4 / 3)\right]^{1/m}$$

$$t_{75} = c\left[-\ln(1-0.75)\right]^{1/m} = c\left[\ln(4)\right]^{1/m}$$
Parameter Estimation for Weibull Distribution

Solve the second equation for $c$ and substitute into the first to get

$$t_{25} = \frac{t_{75} \left[ \ln\left(\frac{4}{3}\right) \right]^{1/m}}{\left[ \ln(4) \right]^{1/m}} = \left[ 0.207519 \right]^{1/m}$$

or

$$m \ln\left( \frac{t_{25}}{t_{75}} \right) = \ln(0.207519) = -1.572534$$

Solve for $m$ directly in terms of the ratio of the two percentiles.

$$\hat{m} = \frac{-1.572534}{\ln\left( \frac{t_{25}}{t_{75}} \right)}$$
Parameter Estimation for Weibull Distribution

With \( m \), we can easily solve for \( c \) using any convenient percentile such as the median, that is,

\[
\hat{c} = \frac{t_{50}}{[\ln(2)]^{1/m}}
\]

How well does this procedure work for fitting a Weibull model to data? To get an answer, let’s try simulation in EXCEL.
Weibull Distribution Simulation in EXCEL

We generate 50 pseudo-random numbers in a column using the EXCEL function =rand(). The adjacent column, based on the inverse CDF calculation, contains the random times distributed according to a Weibull with specified parameters $c$ and $m$.

After estimating the parameters according to the described procedure, we produce a histogram of the data along with a model Weibull fit.
Simulation Results After One Run

Simulation Results

Number

Upper Boundary

0 100 200 300 400 500 600 700 800 900 1000

10 20 30 40 50 60 70 80 90 100

100 200 300 400 500 600 700 800 900 1000

16 14 12 10 8 6 4 2 0
Simulation Results After Second Run

Simulation Results

Upper Boundary

Number

100 200 300 400 500 600 700 800 900 1000
Adding Simulation Runs

By making additional simulation runs, and collecting results from each simulation, we can determine many facets of the problem, such as:

- the sampling distribution of the estimates
- confidence intervals on the estimates
- the suitability of the Weibull model based on these estimates
Simulation for Repairable Systems

- Need to generate random failure events associated with repairable processes.
- Let’s first develop some definitions and concepts for repairable systems.
A system is *repairable* if it can be restored to satisfactory operation by any action, including replacement of components, changes to adjustable settings, swapping of parts, or even a sharp blow with a hammer.
Renewal Process

- Special case of repairable system
- Times between failures are independent and identically distributed (i.i.d.), that is, the times come from a single population
- No trend
Analysis of a Renewal Process

- Reliability analysis methods for non-repairable components have applicability.
- Times between failures may be combined into a single group for analysis.
Homogeneous Poisson Renewal Process

If the times between repairs, $X_i$, are i.i.d. from an exponential distribution with PDF

$$f(x) = \lambda e^{-\lambda x}$$

having constant failure rate $\lambda$, then, the number of repairs $N(t)$ by system age $t$, has a Poisson distribution with mean $M(t) = \lambda t$.

Thus, the probability of observing exactly $N(t) = k$ failures in the interval $(0,t)$ is

$$P[N(t) = k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$
Mean Cumulative Repair Function

- The mean cumulative repair function, $M(t)$, is the average or expected number of repairs for $N(t)$ in time $t$.
- For a HPP, the rate of repairs has a constant value equal to $\lambda$. 
Simulating a Renewal Process

- Specify the distribution
- Generate the random variables from the distribution via simulation
- Arrange the failure times sequentially
- The consecutive, running sums of the failure times represent the system age at each failure point
Simulating a Renewal Process: NHPP Example

- Assume we have a NHPP with the mean time between repairs (MTBF) of 100 hours.
- Let’s see what random outcomes are possible for 20 failures in the life of a system.
- To simulate this process, we substitute twenty unit uniform random numbers, $U$, in the expression:

\[ T = -100 \times \ln U \]
Simulation of HPP in EXCEL

Use =RAND() for U.

<table>
<thead>
<tr>
<th>U</th>
<th>Xi</th>
<th>System Age T</th>
<th>No. Repairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.300</td>
<td>120.3</td>
<td>120.3</td>
<td>1</td>
</tr>
<tr>
<td>0.505</td>
<td>68.4</td>
<td>188.7</td>
<td>2</td>
</tr>
<tr>
<td>0.821</td>
<td>19.7</td>
<td>208.4</td>
<td>3</td>
</tr>
<tr>
<td>0.187</td>
<td>167.5</td>
<td>375.9</td>
<td>4</td>
</tr>
<tr>
<td>0.265</td>
<td>132.7</td>
<td>508.7</td>
<td>5</td>
</tr>
<tr>
<td>0.214</td>
<td>153.9</td>
<td>662.6</td>
<td>6</td>
</tr>
<tr>
<td>0.807</td>
<td>21.4</td>
<td>684.1</td>
<td>7</td>
</tr>
<tr>
<td>0.288</td>
<td>124.5</td>
<td>808.6</td>
<td>8</td>
</tr>
<tr>
<td>0.601</td>
<td>50.9</td>
<td>859.5</td>
<td>9</td>
</tr>
<tr>
<td>0.775</td>
<td>25.5</td>
<td>885.0</td>
<td>10</td>
</tr>
<tr>
<td>0.632</td>
<td>46.0</td>
<td>931.0</td>
<td>11</td>
</tr>
<tr>
<td>0.041</td>
<td>319.1</td>
<td>1250.1</td>
<td>12</td>
</tr>
<tr>
<td>0.150</td>
<td>189.6</td>
<td>1439.7</td>
<td>13</td>
</tr>
<tr>
<td>0.471</td>
<td>75.4</td>
<td>1515.1</td>
<td>14</td>
</tr>
<tr>
<td>0.650</td>
<td>43.1</td>
<td>1558.2</td>
<td>15</td>
</tr>
<tr>
<td>0.026</td>
<td>366.1</td>
<td>1924.3</td>
<td>16</td>
</tr>
<tr>
<td>0.723</td>
<td>32.5</td>
<td>1956.7</td>
<td>17</td>
</tr>
<tr>
<td>0.990</td>
<td>1.0</td>
<td>1957.7</td>
<td>18</td>
</tr>
<tr>
<td>0.929</td>
<td>7.4</td>
<td>1965.1</td>
<td>19</td>
</tr>
<tr>
<td>0.580</td>
<td>54.5</td>
<td>2019.6</td>
<td>20</td>
</tr>
</tbody>
</table>

MTBF 101.0
Plot of One HPP Simulation Run

Cumulative Repairs vs System Age
Two More HPP Simulation Runs (MTBF = 100)
Non-Renewal Repair Process

- The times between failures are not from a single distribution
- The times between failures may not be independent
- There may be trend present, that is, the times between repairs may be growing (system reliability getting better) or decreasing (system reliability getting worse).
Modeling a Non-Renewal Repair Process

- We need to specify a model for simulation.
- We can compare model results to any observed repair history.
- A popular model is a generalization of the HPP. Called the non-homogeneous Poisson process and denoted NHPP, the rate of repairs or intensity $\lambda$ becomes a function $\lambda(t)$ of system age $t$. 
The question for a NHPP is what form should the intensity function $\lambda(t)$ [and consequently $M(t)$] take?

Consider a NHPP with mean repair function modeled by a power relation

$$M(t) = at^b.$$ 

The intensity function is

$$\lambda(t) = \frac{dM(t)}{dt} = abt^{b-1}$$

Note for $b > 1$, the repair rate is increasing in time (system degrading). For $b < 1$, the repair rate decreases (system improvement).
Caution on Terminology

Although the previous expression looks like the hazard rate for a Weibull distribution, leading some to refer to this NHPP as a Weibull process, there is no relation to a Weibull distribution.
Simulating a Power Relation NHPP

- General approach presented in Ross, *Introduction to Probability Models*.
Simulating a NHPP with Power Relation Intensity

Formulas

System age to first repair

\[ Y_1 = \left[ -\frac{1}{a} \ln U_1 \right]^\frac{1}{b} \]

System age to second repair

\[ Y_2 = \left[ Y_1^b - \frac{1}{a} \ln U_2 \right]^\frac{1}{b} \]

System age to \( i \)th repair

\[ Y_i = \left[ Y_{i-1}^b - \frac{1}{a} \ln U_i \right]^\frac{1}{b} \]
Simulation Studies

- Check adequacy and variation in parameter estimation formulas
- Plot confidence intervals along with model estimates
- Try different representations of displaying results
- Experiment with new estimation formulas
- Reveal weaknesses in understanding NHPP processes
- Show range of variability in possible system behavior
# Simulation of Power Relation NHPP in EXCEL

<table>
<thead>
<tr>
<th>Simulation of NHPP with intensity ab t^(b-1)</th>
<th>MLE Estimates (Failure Truncate)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a 0.1</td>
<td>0.1530</td>
</tr>
<tr>
<td>b 0.5</td>
<td>0.4580</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Power Relation Model</th>
<th>Model</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>t</td>
<td>N</td>
</tr>
<tr>
<td>X0 0.305861</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>X1 0.380877</td>
<td>93</td>
<td>1</td>
</tr>
<tr>
<td>X2 0.806257</td>
<td>139</td>
<td>2</td>
</tr>
<tr>
<td>X3 0.463942</td>
<td>380</td>
<td>3</td>
</tr>
<tr>
<td>X4 0.581419</td>
<td>620</td>
<td>4</td>
</tr>
<tr>
<td>X5 0.327763</td>
<td>1301</td>
<td>5</td>
</tr>
<tr>
<td>X6 0.173829</td>
<td>2869</td>
<td>6</td>
</tr>
<tr>
<td>X7 0.386943</td>
<td>3976</td>
<td>7</td>
</tr>
<tr>
<td>X8 0.376467</td>
<td>5303</td>
<td>8</td>
</tr>
<tr>
<td>X9 0.490924</td>
<td>6390</td>
<td>9</td>
</tr>
<tr>
<td>X10 0.575409</td>
<td>7304</td>
<td>10</td>
</tr>
</tbody>
</table>
Example of Power Relation
NHPP Simulation Runs

Intensity Function

MLE Lambda and CI
Summary

- Various ways of simulating data for specific distributions, renewal processes, and non-homogeneous Poisson processes
- Examples of applying simulation techniques to understand theory and check approaches to solving problems
- Power of EXCEL to facilitate simulation and analysis
Where to get more information

- Ross, S., *A Course in Simulation*, (1990), Macmillan